The non-linear Schrödinger equation and the conformal properties of non-relativistic space-time

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Abstract

The cubic non-linear Schrödinger equation where the coefficient of the nonlinear term is a function $F(t,x)$ only passes the Painlevé test of Weiss, Tabor, and Carnevale only for $F=(a+bt)^{-1}$, where a and b are constants. This is explained by transforming the time-dependent system into the constant-coefficient NLS by means of a time-dependent non-linear transformation, related to the conformal properties of non-relativistic space-time. A similar argument explains the integrability of the NLS in a uniform force field or in an oscillator background.

The recent upsurge of interest in non-relativistic conformal symmetries [1, 2, 3, 4] directed attention to their role in getting a deeper understanding, and in physical applications [1]. In this Note we add another example to the list. To be specific, we explain some interesting properties of the non-linear Schrödinger equation (NLS) using these symmetries.

1 The NLS with a position and time-dependent non-linearity

Le us study the cubic NLS

$$iu_t + u_{xx} + F(t,x)|u|^2u = 0, \tag{1.1}$$

where $u=u(t,x)$ is a complex function in $1+1$ space-time dimension. Such an equation arises, for example, in some approaches to the Quantum Hall Effect [5].

When $F(t,x)$ is a constant, this is the usual NLS, which is known to be integrable. But what happens, when the coefficient $F(t,x)$ is a *function* rather then just a constant ?

A useful test of integrability is provided by the *Painlevé test of Weiss, Tabor and Carnevale* [6]. (The procedure is reminiscent of the Frobenius' method used for ODEs).

Let us recall the definition and some properties. For a full account, the Reader is advised to consult [7]. Consider a system of partial differential equations (PDEs), and let us assume

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that its solutions are given by a meromorphic function of the complex veriables $z_1, \dots, z_n$. The singularities of such a function belong to a manifold (called the singular manifold) of dimensions $2n-2$, given by equations of the form $\Phi(z_1, \dots, z_n) = 0$, where the Φ are analytical.

Then our PDE is said to have the *Painlevé property* if all of its solutions can be written, in a neighbourhood of the singular manifold, as a generalized Laurent series,

$$u(z_1, \dots, z_n) = \Phi^\alpha \sum_{j=0}^{\infty} u_j(z_1, \dots, z_n)\Phi^j, \tag{1.2}$$

where α is a negative integer and the $u_j(z_1, \dots, z_n)$ s are analytical. Then the Painlevé conjecture of WTC [6] says that *a PDE which has the Painlevé property is integrable* i.e. can be solved by inverse scattering.

Inserting the expansion (1.2) into our PDE fixes the value of α, and then provides us with recurrence relations for the functions u_j. For some value of j called resonances, u_j remains undetermined, and the system has to satisfy consistency conditions.

Truncating the series may provide us with a Bäcklund transformation [7]. For example, one can generate Jackiw-Pi vortex solutions from the vacuum [8].

Returning to the NLS, below we show

Theorem1 : *The generalized non-linear Schrödinger equation (1.1) only passes the Painlevé test of Weiss, Tabor and Carnevale* [6] *if the coefficient of the non-linear term is*

$$F(t, x) = \frac{1}{a + bt}, \qquad a, b = \text{const}. \tag{1.3}$$

Proof. As it is usual in studying non-linear Schrödiger-type equations [7, 9], we consider Eqn. (1.1) together with its complex conjugate ($v = u^*$),

$$\begin{aligned} iu_t + u_{xx} + Fu^2 v &= 0, \\ -iv_t + v_{xx} + Fv^2 u &= 0. \end{aligned} \tag{1.4}$$

This system will pass the Painlevé test if u et v have generalised Laurent series expansions,

$$u = \sum_{n=0}^{+\infty} u_n \xi^{n-p}, \qquad v = \sum_{n=0}^{+\infty} v_n \xi^{n-q}, \tag{1.5}$$

($u_n \equiv u_n(x,t)$, $v_n \equiv v_n(x,t)$ and $\xi \equiv \xi(x,t)$) in the neighbourhood of the singular manifold $\xi(x,t) = 0$, $\xi_x \neq 0$, with a sufficient number of free coefficients. Owing to a results of Weiss, and of Tabor [7, 10], it is enough to consider $\xi = x + \psi(t)$. Then u_n and v_n become functions de t alone, $u_n \equiv u_n(t)$, $v_n \equiv v_n(t)$. Checking the dominant terms, $u \sim u_0 \xi^{-p}$, $v \sim v_0 \xi^{-q}$, using the above remark, we get

$$p = q = 1, \qquad F\, u_0 v_0 = -2. \tag{1.6}$$

Hence F can only depend on t. Now inserting the developments (1.5) of u and v into (1.4), the terms in ξ^k, $k \geq -3$ read

$$\begin{aligned} &i\Big(u_{k+1,t} + (k+1)u_{k+2}\xi_t\Big) + (k+2)(k+1)u_{k+3} + F\Big(\textstyle\sum_{i+j+l=k+3} u_i u_j v_l\Big) = 0, \\ &i\Big(v_{k+1,t} + (k+1)v_{k+2}\xi_t\Big) - (k+2)(k+1)v_{k+3} - F\Big(\textstyle\sum_{i+j+l=k+3} v_i v_j u_l\Big) = 0. \end{aligned} \tag{1.7}$$

(Condition (1.6) is recovered for $k = -3$). The coefficients u_n, v_n of the series (1.4) are given by the system S_n $(k = n - 3)$,

$$\begin{aligned} &[(n-1)(n-2)-4]u_n + Fu_0^2 v_n = A_n, \\ &Fv_0^2 u_n + [(n-1)(n-2)-4]v_n = B_n, \end{aligned} \tag{1.8}$$

where A_n et B_n only contain those terms u_i, v_j with $i, j < n$. The determinant of the system is

$$\det S_n = n(n-4)(n-3)(n+1). \tag{1.9}$$

Then (1.4) passes the Painlevé test if, for each $n = 0, 3, 4$, one of the coefficients u_n, v_n can be arbitrary. For $n = 0$, (1.6) implies that this is indeed true either for u_0 or v_0. For $n = 1$ and $n = 2$, the system (1.7)-(1.8) is readily solved, yielding

$$\begin{aligned} &u_1 = -\tfrac{i}{2}u_0\xi_t, \qquad v_1 = \tfrac{i}{2}v_0\xi_t, \\ &6v_0u_2 = iv_{0,t}u_0 + 2iu_{0,t}v_0 - \tfrac{1}{2}u_0v_0(\xi_t)^2, \\ &6u_0v_2 = -iu_{0,t}v_0 - 2iv_{0,t}u_0 - \tfrac{1}{2}u_0v_0(\xi_t)^2. \end{aligned} \tag{1.10}$$

$n = 3$ has to be a resonance; using condition (1.6), the system (1.8) becomes

$$\begin{aligned} -2v_0u_3 - 2u_0v_3 &= A_3v_0, \\ -2v_0u_3 - 2u_0v_3 &= B_3u_0, \end{aligned}$$

which requires $A_3v_0 = B_3u_0$. But using the expressions of A_3 and B_3, with the help of "Mathematica" we find

$$2FA_3 = u_0(F_t\xi_t - F\xi_{tt}), \qquad u_0F^2B_3 = F\xi_{tt} - F_t\xi_t,$$

so that the required condition indeed holds.

$n = 4$ has also to be a resonance; we find, as before,

$$\begin{aligned} 2v_0u_4 - 2u_0v_4 &= A_4v_0, \\ -2v_0u_4 - 2u_0v_4 &= B_4u_0, \end{aligned}$$

enforcing the relation $v_0A_4 = -u_0B_4$. Now using the expressions of v_0, u_1, v_1, u_2, v_2 as functions of u_0, F, u_3, v_3, "Mathematica" yields

$$\begin{aligned} 6u_0F^2A_4 = &\quad -F^2u_{0,t}^2 - 2iu_0^2F^2\xi_t\xi_{tt} + u_0F^2u_{0,tt} + iu_0^2F\xi_t^2F_t - u_0Fu_{0,t}F_t + 2u_0F_t^2 - u_0^2FF_{tt}, \\ 3u_0^3F^3B_4 = &\quad -F^2u_{0,t}^2 - 2iu_0^2F^2\xi_t\xi_{tt} + u_0F^2u_{0,tt} + iu_0^2F\xi_t^2F_t - u_0Fu_{0,t}F_t - 4u_0F_t^2 + 2u_0^2FF_{tt}. \end{aligned}$$

Then our constraint implies that

$$2F_t^2 - FF_{tt} = 0. \tag{1.11}$$

Thus $\left(F^{-1}\right)_{tt} = 0$, so that $F^{-1}(x,t) = a + bt$, as stated.

For $b = 0$, $F(t,x)$ in Eqn. (1.1) is a constant, and we recover the constant-coefficient NLS with its known solutions. For $b \neq 0$, the equation becomes explicitly time-dependent. Assuming, for simplicity, that $a = 0$ and $b = 1$, it reads

$$iu_t + u_{xx} + \frac{1}{t}|u|^2u = 0. \tag{1.12}$$

This equation can also be solved. Generalizing the usual travelling soliton, let us seek, for example, a solution of the form

$$u_0(t,x) = e^{i(x^2/4t - 1/t)}\, f(t,x), \tag{1.13}$$

where $f(t,x)$ is some real function. Inserting the Ansatz (1.13) into (1.12), a routine calculation yields the soliton

$$u_0(t,x) = \frac{e^{i(x^2/4t-1/t)}}{\sqrt{t}}\,\frac{\sqrt{2}}{\cosh\big[x/t + x_0\big]}. \tag{1.14}$$

Interestingly, the steps leading to (1.14) are essentially the same as those met when constructing travelling solitons for the ordinary NLS — and this is not a pure coincidence :

<u>Theorem2</u>.

$$u(t,x) = \frac{1}{\sqrt{t}}\exp\left[\frac{ix^2}{4t}\right] U\big(-1/t, -x/t\big) \tag{1.15}$$

satisfies the time-dependent equation (1.12) if and only if $U(t,x)$ *solves Eqn. (1.1) with* $F = 1$.

This can readily be proved by a direct calculation. Inserting (1.15) into (1.12), we find,

$$iu_t + u_{xx} + \frac{1}{t}|u|^2 u = t^{-5/2}\exp\left[\frac{ix^2}{4t}\right]\Big(iU_t + U_{xx} + |U|^2 U\Big), \tag{1.16}$$

proving our statement.

Our soliton (1.14) constructed above comes in fact from the well-known "standing soliton" solution of the NLS,

$$U_0(t,x) = \frac{\sqrt{2}\,e^{it}}{\cosh[x - x_0]}\ , \tag{1.17}$$

by the transformation (1.15). More general solutions could be obtained starting with the travelling soliton

$$U(t,x) = e^{i(vt-kx)}\frac{\sqrt{2}\,a}{\cosh[a(x+kt)]}, \qquad a = \sqrt{k^2+v}. \tag{1.18}$$

2 Non-relativistic conformal transformations

Where does the formula (1.15) come from ? To explain it, let us remember that the non-linear space-time transformation

$$D : \begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} -1/t \\ -x/t \end{pmatrix} \tag{2.1}$$

has already been met in a rather different context, namely in describing planetary motion when the gravitational "constant" changes inversely with time, as suggested by Dirac [11]. Then one shows that

$$\vec{r}(t) = t\,\vec{r}^{\,*}\big(-1/t\big) \tag{2.2}$$

describes planetary motion with Newton's "constant" varying as $G(t) = G_0 t$, whenever $\vec{r}^{\,*}(t)$ describes ordinary planetary motion, i.e. the one with a constant gravitational constant, $G(t) = G_0$ [12] [1].

The strange-looking transformation (2.1) is indeed related to the conformal structure of non-relativistic space-time [4, 12, 15, 16]. It has been noticed a long time ago [17], that the

[1] Curiously, the *same* transformation is used to transform supernova explosion into implosion, [13, 14].

"conformal" space-time transformations

$$
\begin{cases}
\begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} \delta^2 t \\ \delta\, x \end{pmatrix}, & 0 \neq \delta \in \mathbb{R} \quad \text{dilatations} \\
\begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} \dfrac{t}{1-\kappa t} \\ \dfrac{x}{1-\kappa t} \end{pmatrix}, & \kappa \in \mathbb{R} \quad \text{expansions} \\
\begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} t+\epsilon \\ x \end{pmatrix}, & \epsilon \in \mathbb{R} \quad \text{time translations}
\end{cases}
\tag{2.3}
$$

implemented on wave functions according to

$$
U(T,X) = \begin{cases}
\delta^{1/2} u(t,x) \\
(1-\kappa t)^{1/2} \exp\Big[i\dfrac{\kappa x^2}{4(1-\kappa t)}\Big] u(t,x) \\
u(t,x)
\end{cases}
\tag{2.4}
$$

permute the solutions of the free Schrödinger equation. In other words, they are *symmetries* for the free Schrödinger equation. (The generators in (2.3) span in fact an $\mathrm{SL}(2,\mathbb{R})$ group; when added to the obvious galilean symmetry, the Schrödinger group is obtained. A Dirac monopole, an Aharonov-Bohm vector potential, and an inverse-square potential can also be included, [18, 12, 19]).

The transformation D in Eqn. (2.1) belongs to this symmetry group: it is in fact (i) a time translation with $\epsilon = 1$, (ii) followed by an expansion with $\kappa = 1$, (iii) followed by a second time-translation with $\epsilon = 1$. It is hence a symmetry for the free (linear) Schrödinger equation. Its action on ψ, deduced from (2.4), is precisely (1.15).

The cubic NLS with non-linearity $F = \text{const.}$ is not more $SL(2,\mathbb{R})$ invariant [2]. In particular, the transformation D in (2.1), implemented as in Eq. (1.15) carries the cubic term into the time-dependent term $(1/t)|u|^2u$ — just like Newton's gravitational potential G_0/r with $G_0 = \text{const.}$ is carried into the time-dependent Dirac expression $t^{-1}G_0/r$ [12].

Similar arguments explain the integrability of other NLS-type equations. For example, electromagnetic waves in a non-uniform medium propagate according to

$$
iu_t + u_{xx} + \big(-2\alpha x + 2|u|^2\big)u = 0,
\tag{2.5}
$$

which can again be solved by inverse scattering [21]. This is explained by observing that the potential term here can be eliminated by switching to a uniformly accelerated frame:

$$
\begin{aligned}
&u(t,x) = \exp\big[-i(2\alpha x t + \tfrac{4}{3}\alpha^2 t^3\big] U(T,X),\\
&T = t, \qquad X = x + 2\alpha t^2.
\end{aligned}
\tag{2.6}
$$

Then $u(t,x)$ solves (2.5) whenever $U(T,X)$ solves the free equation $iU_t + U_{xx} + 2|U|^2U = 0$.

The transformation (2.6) is again related to the structure of non-relativistic space-time. It can be shown in fact [10] that the (linear) Schrödinger equation

$$
iu_t + u_{xx} - V(t,x)u = 0
\tag{2.7}
$$

[2] Galilean symmetry can be used to produce further solutions — just like the travelling soliton (1.18) can be obtained from the "standing one" in (1.17) by a galilean boost. Full Schrödinger invariance yielding expanded and dilated solutions can be restored by replacing the cubic non-linear term by the fifth-order non-linearity $|\psi|^4\psi$. These statements about non-invariance assume restricting ourselves to certain representations, see [20].

can be brought into the free form $iU_T + U_{XX} = 0$ by a space-time transformation $(t, x) \to (T, X)$ if and only if the potential is

$$V(t,x) = \alpha(t)x \pm \frac{\omega^2(t)}{4}x^2. \tag{2.8}$$

For the uniform force field ($\omega = 0$) the required space-time transformation is precisely (2.6). For the oscillator potential ($\alpha = 0$), one can use rather Niederer's transformation [22, 19]

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{\cos\omega t}} \exp\big[-i\tfrac{\omega}{4}x^2\tan\omega t\big]\, U(T,X), \\ T &= \frac{\tan\omega t}{\omega} \qquad X = \frac{x}{\cos\omega t}. \end{aligned} \tag{2.9}$$

Then

$$iu_t + u_{xx} - \frac{\omega^2 x^2}{4}u = (\cos\omega t)^{-5/2}\, \exp\big[-i\frac{\omega}{4}\tan\omega t\big]\Big(iU_T + U_{XX}\Big). \tag{2.10}$$

Restoring the nonlinear term allows us to infer that

$$iu_t + u_{xx} + \Big(-\frac{\omega^2 x^2}{4} + \frac{1}{\cos\omega t}\,|u|^2\Big)u = 0 \tag{2.11}$$

is integrable, and its solutions are obtained from those of the "free" NLS by the transformation (2.9).

3 Discussion

To conclude, we us mention some more related results.

Firstly, our result should be compared with the those of Chen et al. [23], who prove that the equation

$$iu_t + u_{xx} + F(|u|^2)u = 0 \tag{3.1}$$

can be solved by inverse scattering if and only if $F(|u|^2) = \lambda|u|^2$, where $\lambda = \text{const}$. Note, however, that Chen et al. only study the case when the functional $F(|u|^2)$ is independent of the space-time coordinates t and x.

It has also been shown that the non-linear Schrödinger equation with time–dependent coefficients,

$$iu_t + p(t)u_{xx} + F(t)|u|^2u = 0, \tag{3.2}$$

can be transformed into the constant–coefficient form whenever [24]

$$p(t) = F(t)\left(a + b\int^t p(s)ds\right). \tag{3.3}$$

This same condition, which could also be obtaind by a suitable generalization of our approach, was found later as the one needed for the Painlevé test [25] applied to Eq. (3.3).

On the other hand, the constant-coefficient, damped, driven NLS,

$$iu_t + u_{xx} + F(t)|u|^2u = a(t,x)u + b(t,x), \tag{3.4}$$

was shown to pass the Painlevé test if

$$a(t,x) = \big(\tfrac{1}{2}\partial_t\beta - \beta^2\big) + i\beta(t) + \alpha_1(t) + \alpha_0(t), \qquad b(t,x) = 0, \tag{3.5}$$

[26], i.e., when the potential can be transformed away by our "non-relativistic conformal transformations".

We only studied the case of $d = 1$ space dimension. Similar results would hold for any $d \geq 1$. It is worth noting that more general dynamical symmetries of the NLS under subalgebras of the Schrödinger/conformal algebra were studied systematically by S. Stoimenov and M. Henkel [20].

At last, it is worth noting that the "Kaluza-Klein-type" framework, first proposed by Duval et al. [15, 12] has attracted considerable recent attraction, namely in the non-relativistic AdS/FCT context. See, fore example, [27].

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